1st Junior Penchick Online Mathematical Olympiad

Qualifying Stage, Paper D

(Ages 16 and below)

9 May 2026

INSTRUCTIONS AND INFORMATION

1. **There are 20 questions to be answered in 90 minutes, split into two parts.** The highest possible score for this contest is 48 points.
   - *Part I (multiple-choice):* Each question has four choices (A, B, C, and D), and only one of these choices is correct. Submit the best answer to each question. Two points are given for a correct answer, for a total of 24 points for this section.
   - *Part II (short answer):* Submit the best answer to each question. Three points are given for a correct answer, for a total of 24 points for this section. For this paper, all answers are **integers** from 0 to 999 inclusive.
2. **Always exhibit academic honesty during the contest.** Do not use calculators, protractors, graph paper, search engines, artificial intelligence, and other external applications, resources, or help while answering the contest. Rulers, compasses, and blank or lined paper are allowed. Cheating in any form is strictly prohibited and will be acted upon based on the violation system.
3. Note that diagrams are **not necessarily drawn to scale**.
4. The top 60% of contestants in this paper will be invited to take the **1st JPOMO Final Stage** on 27 June 2026.

Enjoy the contest!

Part I: Multiple-Choice Questions

Each question has four choices (A, B, C, and D), and only one of these choices is correct. Submit the best answer to each question. Two points are given for a correct answer, for a total of 24 points for this section.

1. Let $a = 2^3 \cdot 3^2 \cdot 5$ and $b = 2 \cdot 3^4 \cdot 7$. Find the greatest common divisor of a and b.

(a) 8 (b) 9 (c) 18 (d) 36

2. The masses, in grams, of a group of penchicks are recorded below:

$$420,\ 435,\ 435,\ 450,\ 460,\ 460,\ 460,\ 475,\ 490$$

Which of the following gives the correct median, mode and range of the masses?

Option	Median (g)	Mode (g)	Range (g)
A	450	460	70
B	460	460	70
C	460	435	55
D	450	435	55

(a) A (b) B (c) C (d) D

3. A triangle has 3 angles which are all prime numbers. What could the largest angle be?

(a) 163 (b) 167 (c) 173 (d) 177

4. How many distinct rearrangements are there of the phrase

PENCHICK ONLINE MATH OLYMPIAD

where spaces are ignored and repeated letters are considered indistinguishable?

(a) $26!$ (b) $\dfrac{26!}{3!\,3!\,(2!)^6}$ (c) $\dfrac{26!}{3!\,3!\,(2!)^7}$ (d) $\dfrac{26!}{3!\,3!\,(2!)^8}$

5. 150 young adults were asked how they commuted to work. 125 said they took the underground train and 93 said they cycled. Of all the people interviewed, 72 said they both cycled and took the underground train. How many people do not cycle and do not take the underground train?

(a) 4 (b) 6 (c) 14 (d) 16

6. A square with side length x and a circle share the same center. The total area of the regions that are inside the circle and outside the square is equal to the total area of the regions that are outside the circle and inside the square. What is the radius of the circle?

(a) $\frac{x}{\sqrt{\pi}}$ (b) $\frac{1+\sqrt{x}}{2}$ (c) $\frac{\sqrt{x+1}}{2}$ (d) $\sqrt{\frac{\pi x}{2}}$

7. How many positive divisors does the number 2026^{2026} have?

(a) 2026 (b) 2026^2 (c) 2027 (d) 2027^2

8. Triangle ABC is isosceles with $AB = AC = 10$ and $BC = 12$. Let D be a point on AC such that BD is perpendicular to AC. Find the length BD.

(a) 6.4 (b) 8 (c) 9.6 (d) 10.8

9. There are 2026 Penchicks playing a game, one of which is King Renren and the rest are knights. They are all sat equally around a table. In each round, every remaining penchick points at exactly one other penchick at random. If a prime number of penchicks, regardless of their rank, all point at the same penchick, again regardless of their rank, then all of those pointing penchicks are eliminated from the game. Regardless of if any penchicks are eliminated or not, another round starts, wherein the elimination process continues over and over until only one or two penchicks remain. If only two penchicks remain, then the remaining two play rock-paper-scissors at random over and over until a winner, who will be the only one remaining, is chosen. What is the probability that the final remaining penchick is King Renren?

(a) $\frac{1}{2025}$ (b) $\frac{1}{2026}$ (c) $\frac{1}{2027}$ (d) Other

10. The polynomials x^2+ax+b and x^2+bx+a share exactly one root. if $a \neq b$, then what is the value of $a+b$?

(a) -2 (b) -1 (c) 0 (d) 1

11. A Grab driver is at point $(3,0)$ and needs to deliver food to a house at point $(9,5)$. However, before reaching the house, the driver must stop at a McDonald's located somewhere on the y-axis. Assuming the driver travels in straight lines, what is the shortest total distance the driver can travel?

(a) 11 (b) 12 (c) 13 (d) 14

12. Let α, β, γ be the roots of the cubic equation

$$x^3 - 7x^2 + 3x + 2 = 0.$$

Find the cubic equation whose roots are

$$\alpha + \beta, \quad \alpha + \gamma, \quad \beta + \gamma.$$

(a) $2x^3 - 7x^2 + 52x - 23$

(b) $x^3 - 14x^2 + 52x - 23$

(c) $2x^3 + 14x^2 - 26x + 23$

(d) $x^3 + 7x^2 + 13x + 23$

Part II: Short-Answer Questions

Submit the best answer to each question. Three points are given for a correct answer, for a total of 24 points for this section.
For this paper, all answers are **integers** from 0 to 999 inclusive.

1. Penchick is thinking of a two-digit number. He tells Pelican the tens digit and Blizzy the ones digit, and challenges them to guess the number. Their conversation goes as follows:

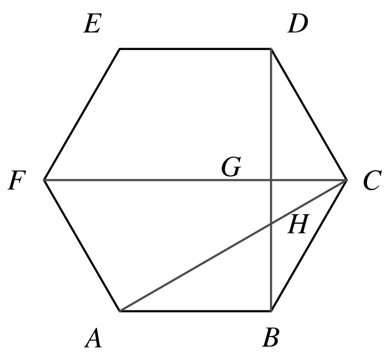
- Blizzy: I know the number is not a multiple of 5.
- Penchick: Correct, it is a prime number.
- Pelican: Now I know what the number is.

What was the two-digit number Penchick think about?

2. If real numbers x, y are there such that $x^2 + y^2 - 34x - 34y + 109 = 0$ and $xy = -90$, then find the remainder when $x^3 + y^3$ is divided by 1000.

3. Pentrick has 4 red balls, 5 blue balls, and 6 green balls. He randomly picks 3 balls. The probability that at least two of the balls share the same color can be written as $\frac{x}{y}$ in lowest terms. Find $x + y$.

4. $ABCDEF$ is a regular hexagon. Diagonals AC and DB intersect at H, and FC intersects DB at G. The quadrilateral $AFGH$ is formed. If the area of quadrilateral $AFGH$ is $\frac{a}{b}$ of the area of the hexagon, where a and b are coprime positive integers, find $a + b$.



5. Penrick writes the following huge expression on the whiteboard:

$$37^{500} + 7^4 + 67 \times 360 - 85$$

If the remainder of the expression when divided by 18 and 12 are m and n, respectively, then find $50m + n$.

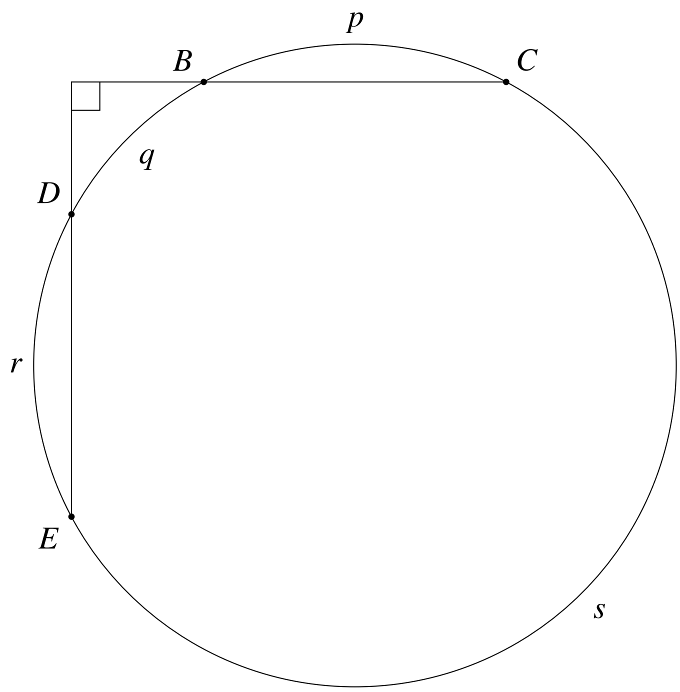
6. If a, b, c, d are the roots of $3x^4 - 2x^3 + 5x^2 - x + 4$, find $(a-1)(b-1)(c-1)(d-1)$.

7. The 12 Chrysos Heirs of Amphoreus are seated around a round table. All heirs are distinct. Due to internal conflicts, the following conditions must be satisfied:

- Anaxa and Aglaea must sit directly opposite each other.
- Hyacine must sit adjacent to Aglaea.
- Phainon and Mydei must not sit next to each other.

Let N be the total number of distinct seating arrangements that satisfy these conditions. Find the remainder when N is divided by 1000.

8. In the figure, the circle is divided into four arcs. Let $\overset{\frown}{BC} = p$, $\overset{\frown}{BD} = q$, $\overset{\frown}{DE} = r$, and the remaining arc $\overset{\frown}{s}$. Given that $p = 6$ cm, $q = 2$ cm, $r = 12$ cm, find s in cm. *(Take note that the diagram is not to scale)*



Answer Key

Part I: Multiple-Choice Questions

1. c	**5.** a	**9.** b
2. b	**6.** a	**10.** b
3. c	**7.** d	**11.** c
4. d	**8.** c	**12.** b

Part II: Short-Answer Questions

1. 97	**5.** 651 $(m = 13, n = 1)$
2. 503	**6.** 3
3. 158 $(x = 67, y = 91)$	**7.** 640 $(N = 584,640)$
4. 47 $(a = 11, b = 36)$	**8.** 24

Solutions

Part I: Multiple-Choice Questions

1. We have $a = 2^3 \cdot 3^2 \cdot 5$ and $b = 2 \cdot 3^4 \cdot 7$. To find the greatest common divisor, we take the common prime factors with the smaller exponents. Thus,

$$\gcd(a, b) = 2^{\min(3,1)} \cdot 3^{\min(2,4)} = 2^1 \cdot 3^2 = 2 \cdot 9 = 18.$$

Therefore, the greatest common divisor of a and b is $\boxed{\text{(c) } 18}$.

2. The masses are already arranged in ascending order:

$$420,\ 435,\ 435,\ 450,\ 460,\ 460,\ 460,\ 475,\ 490.$$

Since there are 9 values, the median is the 5^{th} value, which is 460 g. The mode is 460 g because it appears most often. The range is $490 - 420 = 70$ g. Therefore, the correct option is $\boxed{\text{(b) B}}$.

3. The angles of a triangle sum to $180°$. Since all three angles are prime numbers, one angle must be $2°$, the only even prime, because the sum of three odd primes would be odd. To maximize the largest angle, we minimize the other two prime angles. Using $2°$ and $3°$ gives $180 - 2 - 3 = 175°$, but 175 is not prime. Using $2°$ and $5°$ gives $180 - 2 - 5 = 173°$, which is prime. Therefore, the largest possible angle is $\boxed{\text{(c) } 173°}$.

4. Ignoring spaces, the phrase PENCHICK ONLINE MATH OLYMPIAD contains 26 letters. The letters I and N each appear 3 times, while the letters A, C, E, H, L, M, O, and P each appear 2 times. Therefore, the number of distinct rearrangements is

$$\boxed{\text{(d) } \frac{26!}{3!\,3!\,(2!)^8}}.$$

5. To find the number of people who do not cycle and do not take the underground, we can first find the number of people who do cycle or take the underground. A first count gives $125+93=218$ people who do cycle or take the underground. However, there is an overlap of 72, which we counted twice when we added the two numbers. Therefore, we should subtract 72 from 218 once to obtain $218-72=146$ people who commute in one of the two ways. Therefore, we have $150-146=\boxed{\text{(a) } 4}$ people who use neither method of transportation.

6. Let the region within the circle and square be a. In other words, it is the area inside the circle and the square. Let r be the radius. We know that the area of the circle minus a is equal to the area of the square, minus a. We get:

$$\pi r^2 - a = x^2 - a$$

$$r^2 = \frac{x^2}{\pi}$$

$$r = \boxed{\text{(a) } \frac{x}{\sqrt{\pi}}}$$

7. First, factorize 2026, which leads to $2026 = 1013 \cdot 2$. Notice that 1013 is prime. Hence,

$$2026^{2026} = (2 \cdot 1013)^{2026} = 2^{2026} \cdot 1013^{2026}.$$

The total number of positive divisors is obtained by adding 1 to each exponent and multiplying:

$$(2026+1)(2026+1) = \boxed{\text{(d) } 2027^2}.$$

8. Let $AD = x$, so $DC = 10-x$. Since $BD \perp AC$, triangles ABD and CBD are right triangles. By the Pythagorean Theorem, we get the two following equations:

$$AB^2 = AD^2 + BD^2 \implies 100 = x^2 + BD^2 \tag{1}$$

$$BC^2 = DC^2 + BD^2 \implies 144 = (10-x)^2 + BD^2 \tag{2}$$

Notice that subracting Equation 1 from Equation 2 removes BD and leaves x, which results in:

$$144 - 100 = (10-x)^2 - x^2.$$

So,

$$44 = 100 - 20x + x^2 - x^2 = 100 - 20x,$$

Hence,

$$20x = 56 \quad \Rightarrow \quad x = 2.8.$$

Now substitute into $100 = x^2 + BD^2$:

$$BD^2 = 100 - 2.8^2 = 100 - 7.84 = 92.16.$$

Therefore,

$$BD = \sqrt{92.16} = \boxed{\text{(c) } 9.6}.$$

9. Let P_i be the probability that penchick i is the final winner. The game rules treat every penchick identically: no one has any advantage over any other, and every choice is made randomly in the same way. Therefore, by symmetry,

$$P_1 = P_2 = \cdots = P_{2026}.$$

Also, exactly one penchick must be the final winner, so the probabilities of all 2026 penchicks add up to 1:

$$P_1 + P_2 + \cdots + P_{2026} = 1.$$

Hence,

$$2026P_1 = 1 \quad \Longrightarrow \quad P_1 = \frac{1}{2026}.$$

Since King Renren is just one of the 2026 penchicks, the probability that he is the last remaining penchick is

$$\boxed{\text{(b) } \frac{1}{2026}}.$$

10. Let the common root be r. Then $r^2 + ar + b = 0$ and $r^2 + br + a = 0$. After subtraction, the equation becomes:

$$\begin{aligned} (a-b)r + (b-a) &= 0 \\ (a-b)r - (a-b) &= 0 \\ (a-b)(r-1) &= 0 \end{aligned}$$

Since $a \neq b$, we must have $r = 1$. Substituting into either equation,

$$1 + a + b = 0 \implies a + b = -1$$

Therefore, the answer is $\boxed{\text{(b) } -1}$.

11. Let the McDonald's be at $(0, y)$ on the y-axis. Then the total distance is:

$$d(y) = \sqrt{(3-0)^2 + (0-y)^2} + \sqrt{(9-0)^2 + (5-y)^2}.$$

So,

$$d(y) = \sqrt{9 + y^2} + \sqrt{81 + (5-y)^2}.$$

To minimize this, reflect the house $(9,5)$ across the y-axis to get $(-9,5)$. Then the shortest broken path from $(3,0)$ to $(9,5)$ through the y-axis is equal to the straight-line distance from $(3,0)$ to $(-9,5)$:

$$\sqrt{(3-(-9))^2 + (0-5)^2} = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = \boxed{\text{(c) } 13}$$

12. By Vieta's formulas, if the roots of $x^3 - 7x^2 + 3x + 2 = 0$ are α, β, γ, then

$$\alpha + \beta + \gamma = 7, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = 3, \qquad \alpha\beta\gamma = -2.$$

Now let the roots of the new polynomial be

$$u_1 = \alpha + \beta, \qquad u_2 = \alpha + \gamma, \qquad u_3 = \beta + \gamma.$$

To find the polynomial with roots u_1, u_2, u_3, we use Vieta's formulas in reverse. Since the polynomial is monic, it must have the form

$$x^3 - (u_1 + u_2 + u_3)x^2 + (u_1u_2 + u_2u_3 + u_3u_1)x - u_1u_2u_3 = 0.$$

So we only need to find the sum, the sum of pairwise products, and the product of the new roots. First, compute the sum:

$$\begin{aligned} u_1 + u_2 + u_3 &= (\alpha+\beta) + (\alpha+\gamma) + (\beta+\gamma) \\ &= 2(\alpha+\beta+\gamma) \\ &= 2(7) = 14. \end{aligned}$$

Next, compute the sum of the pairwise products:

$$u_1u_2 + u_2u_3 + u_3u_1 = (\alpha+\beta)(\alpha+\gamma) + (\alpha+\gamma)(\beta+\gamma) + (\beta+\gamma)(\alpha+\beta).$$

Now expand each product:

$$\begin{aligned} (\alpha+\beta)(\alpha+\gamma) &= \alpha^2 + \alpha\gamma + \alpha\beta + \beta\gamma, \\ (\alpha+\gamma)(\beta+\gamma) &= \alpha\beta + \alpha\gamma + \beta\gamma + \gamma^2, \\ (\beta+\gamma)(\alpha+\beta) &= \alpha\beta + \alpha\gamma + \beta^2 + \beta\gamma. \end{aligned}$$

Adding these expressions gives

$$u_1u_2 + u_2u_3 + u_3u_1 = (\alpha^2 + \beta^2 + \gamma^2) + 3(\alpha\beta + \beta\gamma + \gamma\alpha).$$

Now find $\alpha^2 + \beta^2 + \gamma^2$ using the formula $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$. Substituting the known values from above leads to $\alpha^2 + \beta^2 + \gamma^2 = 7^2 - 2(3) = 49 - 6 = 43$. So,

$$\begin{aligned} u_1u_2 + u_2u_3 + u_3u_1 &= 43 + 3(3) \\ &= 43 + 9 \\ &= 52. \end{aligned}$$

Finally, compute the product:

$$u_1u_2u_3 = (\alpha+\beta)(\alpha+\gamma)(\beta+\gamma).$$

A useful identity is

$$(\alpha+\beta)(\alpha+\gamma)(\beta+\gamma) = (\alpha+\beta+\gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma,$$

which can be verified by expanding the left side and collecting terms. Substituting the values from Vieta's formulas gives

$$u_1u_2u_3 = 7(3) - (-2) = 21 + 2 = 23.$$

Therefore, the cubic with roots u_1, u_2, u_3 after substituting the values into $x^3 - (u_1 + u_2 + u_3)x^2 + (u_1u_2 + u_2u_3 + u_3u_1)x - u_1u_2u_3 = 0$ is:

$$\boxed{\text{(b) } x^3 - 14x^2 + 52x - 23 = 0}.$$

Part II: Short-Answer Questions

1. Blizzy knows only the ones digit of the number. A number is a multiple of 5 precisely when its ones digit is 0 or 5, so Blizzy can state with certainty that the number is not a multiple of 5 only when the ones digit is neither 0 nor 5. Blizzy's statement therefore reveals to the others that the ones digit is not 0 or 5.

   After Penchick confirms the number is prime, the candidates become all two-digit primes. Since no two-digit prime ends in 0 or 5, Blizzy's clue eliminates none of them from consideration — but it does confirm the ones-digit restriction to Pelican. Pelican, who knows the tens digit and now knows the number is a prime with ones digit not 0 or 5, can identify the number uniquely. Checking the 2-digit prime numbers by tens digit, we have:

$$\begin{aligned}
1&:\ 11,13,17,19\\
2&:\ 23,29\\
3&:\ 31,37\\
4&:\ 41,43,47\\
5&:\ 53,59\\
6&:\ 61,67\\
7&:\ 71,73,79\\
8&:\ 83,89\\
9&:\ 97
\end{aligned}$$

   Only the tens digit 9 corresponds to exactly one prime, so the number is $\boxed{97}$.

2. Notice that $x^3+y^3=(x+y)(x^2-xy+y^2)$, so it suffices to find $x+y$. Let $u=x+y$. We rewrite x^2+y^2 in terms of u:

$$x^2+y^2=(x+y)^2-2xy=u^2-2(-90)=u^2+180.$$

   Substituting into the original equation $x^2+y^2-34x-34y+109=0$ gives:

$$(u^2+180)-34u+109=0 \implies u^2-34u+289=0 \implies (u-17)^2=0,$$

   so $u=x+y=17$. We then compute:

$$x^2-xy+y^2=(x+y)^2-3xy=289-3(-90)=289+270=559,$$

   and therefore:

$$x^3+y^3=(x+y)(x^2-xy+y^2)=17\times 559=9503.$$

   Therefore, the answer is the last three digits of x^3+y^3, which is $\boxed{503}$.

3. We use complementary counting. The total number of ways to choose 3 balls from $4+5+6=15$ is $\binom{15}{3}=455$. The complementary event is picking one ball of each colour: there are 4 choices for the red ball, 5 for the blue, and 6 for the green, giving $4\times5\times6=120$ selections with all different colours. Hence, the number of selections with at least two balls of the same colour is $455-120=335$, and the probability is

$$\frac{335}{455}=\frac{67}{91}.$$

   Since $\gcd(67,91)=1$, the fraction is in lowest terms, giving $x+y=67+91=\boxed{158}$.

4. Let the side length of the hexagon be $2r$, so the apothem (the distance from the centre O to any side) is $r\sqrt{3}$. Partitioning the hexagon into six congruent triangles from the centre, each has base $2r$ and height $r\sqrt{3}$, giving

$$\text{Area of } ABCDEF = 6\cdot\tfrac{1}{2}(2r)(r\sqrt{3}) = 6r^2\sqrt{3}.$$

Vertices C and F are opposite, so diagonal CF passes through O and haslength $4r$. Since $BC = CD = 2r$, triangle BCD is isosceles; the axis of symmetry of the hexagon through C and F bisects BD perpendicularly, so CF and BD are perpendicular, and we call their intersection point G. Vertex A is adjacent to F, so the perpendicular distance from A to the line CF equals the apothem of the hexagon, which is $r\sqrt{3}$. Hence

$$\text{Area of } \triangle AFC = \tfrac{1}{2}\cdot 4r\cdot r\sqrt{3} = 2r^2\sqrt{3}.$$

Triangle GCH is right-angled at G, since G is the intersection of the two perpendicular lines CF and DB. Since G is the midpoint of OC (as DB passes through the point on CF at distance r from O), the leg $GC = r$. The diagonal AC meets CF at an angle of 30° at C, which is a standard angle in the regular hexagon, so the other leg satisfies $GH = GC\tan 30^\circ = \dfrac{r}{\sqrt{3}}$. Therefore,

$$\text{Area of } \triangle GCH = \tfrac{1}{2}\cdot GC\cdot GH = \tfrac{1}{2}\cdot r\cdot \frac{r}{\sqrt{3}} = \frac{r^2}{2\sqrt{3}}.$$

The quadrilateral $AFGH$ is obtained by removing triangle GCH from triangle AFC:

$$\text{Area of } AFGH = 2r^2\sqrt{3} - \frac{r^2}{2\sqrt{3}} = \frac{12r^2\sqrt{3}}{6} - \frac{r^2\sqrt{3}}{6} = \frac{11r^2\sqrt{3}}{6}.$$

Dividing by the area of the hexagon,

$$\frac{\text{Area of } AFGH}{\text{Area of } ABCDEF} = \frac{\frac{11}{6}r^2\sqrt{3}}{6r^2\sqrt{3}} = \frac{11}{36}.$$

Since $a = 11$ and $b = 36$ are coprime positive integers, $a + b = \boxed{47}$.

5. We take the expression $\pmod{18}$ and $\pmod{12}$, first starting with $\pmod{18}$. Since working $\pmod{18}$ means we only care about the remainder when dividing by 18, we may replace numbers with equivalent remainders of $\pmod{18}$. Notice that:

$$\begin{aligned} 37 &\equiv 1 \pmod{18}, \\ 360 &\equiv 0 \pmod{18}. \end{aligned}$$

From this information, we can simplify the original expression as:

$$\begin{aligned} 37^{500} + 7^4 + 67\times 360 - 85 &\equiv 1^{500} + 7^4 + 67\cdot 0 - 85 \pmod{18} \\ &\equiv 1 + 7^4 - 85 \pmod{18}. \end{aligned}$$

Now, instead of computing 7^4 directly, we note that:

$$7^2 = 49 \equiv 13 \pmod{18} \implies 7^4 \equiv 13^2 = 169 \equiv 7 \pmod{18}.$$

Hence,

$$\begin{aligned}1+7^4-85 &\equiv 1+7-85 \pmod{18}\\ &\equiv -77 \pmod{18}\\ &\equiv 13 \pmod{18}.\end{aligned}$$

For the remainder when divided by 12, we proceed similarly. Since

$$\begin{aligned}37 &\equiv 1 \pmod{12},\\ 360 &\equiv 0 \pmod{12},\\ 7^2 &\equiv 1 \pmod{12}.\end{aligned}$$

we have $7^4 \equiv 1 \pmod{12}$, giving:

$$\begin{aligned}37^{500}+7^4+67\times 360-85 &\equiv 1+7^4+0-85 \pmod{12}\\ &\equiv 1+1-85 \pmod{12}\\ &\equiv 1 \pmod{12}.\end{aligned}$$

Thus, $m = 13$ and $n = 1$. As such, the answer is $13\times 50+1 = \boxed{651}$.

6. Let $P(x) = 3x^4-2x^3+5x^2-x+4$. Since a,b,c,d are the roots, we can write $P(x) = 3(x-a)(x-b)(x-c)(x-d)$. Substituting $x = 1$:

$$P(1) = 3(1-a)(1-b)(1-c)(1-d).$$

Since there are four factors, the negatives cancel, so $(1-a)(1-b)(1-c)(1-d) = (a-1)(b-1)(c-1)(d-1)$, giving

$$(a-1)(b-1)(c-1)(d-1) = \frac{P(1)}{3}.$$

Computing $P(1) = 3-2+5-1+4 = 9$, we get

$$(a-1)(b-1)(c-1)(d-1) = \boxed{3}.$$

7. Since rotations are identical, fix Anaxa at seat 1. With 12 equally-spaced seats, the seat directly opposite is seat 7, so Aglaea occupies seat 7. Hyacine must sit adjacent to Aglaea, so Hyacine occupies seat 6 or seat 8: two choices.

In the case Hyacine $=$ seat 6, the three fixed seats $\{1,6,7\}$ divide the remaining 9 seats into two arcs: seats $\{2,3,4,5\}$ between Anaxa (seat 1) and Hyacine (seat 6), forming an arc of 4 seats, and seats $\{8,9,10,11,12\}$ between Aglaea (seat 7) and Anaxa (seat 1) going the other way, forming an arc of 5 seats. The case Hyacine $=$ seat 8 is symmetric, yielding arcs of 5 and 4 in the opposite positions. In both cases, the two arcs share no adjacencies across the fixed trio, and the number of adjacent pairs within them is $(4-1)+(5-1) = 7$, since a chain of k seats contains exactly $k-1$ adjacent pairs. For each such adjacent pair, Phainon and Mydei can be arranged in 2 ways, and the remaining 7 heirs fill the rest in $7!$ ways, giving $7\times 2\times 7!$ invalid arrangements per choice of Hyacine. Hence the valid count for one choice is $9!-14\cdot 7!$, and summing over both choices,

$$N = 2(9!-14\cdot 7!) = 2(362880-70560) = 2\times 292320 = 584640.$$

The remainder when N is divided by 1000 is $\boxed{640}$.

8. Let the central angles subtending arcs $\overset{\frown}{BC}$, $\overset{\frown}{BD}$, $\overset{\frown}{DE}$, and the remaining arc be A, B, C, and D, respectively. Since the four arcs partition the circle,

$$A+B+C+D=360^\circ.$$

From the figure, the geometric configuration produces a right angle of 90°. The two arcs intercepted on opposite sides of this angle are $(A+B)$ and $(B+C)$, each of which spans two consecutive arcs. Since the angle equals half the sum of the two intercepted arcs,

$$\frac{1}{2}\big((A+B)+(B+C)\big)=90^\circ \implies A+2B+C=180^\circ.$$

Subtracting this from the full-circle equation gives

$$D-B=360^\circ-180^\circ=180^\circ.$$

Since arc length is proportional to central angle, the same relation holds for arc lengths. Writing D and B in terms of s and q,

$$s-q=\frac{180^\circ}{360^\circ}(p+q+r+s)=\frac{1}{2}(p+q+r+s).$$

Solving for s:

$$2(s-q)=p+q+r+s \implies s=p+3q+r.$$

Substituting the given values,

$$s=6+3(2)+12=\boxed{24}\text{ cm}.$$